
CHAPTER 3 ***TRANSIENT CIRCUIT ANALYSIS***

3.1. First order transient circuits

Introduction

A first-order circuit is characterized by a first-order differential equation. there are two types of first-order circuits, *RC & RL.*

In addition there being two types of first-order circuits, there are two ways to excite the ckt's:

(I) *By initial conditions of the storage elements in the ckts, called as source free ckts*
(II) *By independent sources.*

*T*he source- free RC circuit

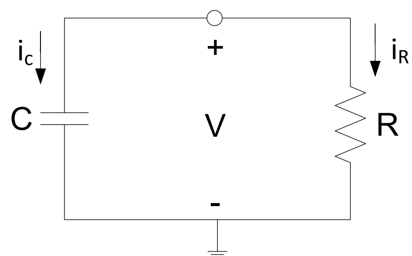


Fig.3.1.1. A source free RC ckt

Our objective is to determine the ***circuit response*** *(A circuit response is the manner in which the circuit reacts to an excitation)*, which, we assume to be the voltage V(t) across the capacitor. Since the capacitor is initially charged, we can assume that at time t = 0, the initial voltage is

$$V(0) = V_0$$

Applying KCL at the top node of the circuit in Fig.3.1.1,

$$i_c + i_R = 0$$

But, $i_c = C\, dv/dt$ and $i_R = v/R$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

Rearranging terms,

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both sides, we get

$$\ln(v) = -\frac{t}{RC} + \ln(A)$$

$$V(t) = Ae^{\frac{-t}{RC}}$$

But from the initial conditions, $V(0) = A = V_0$. Hence,

$$V(t) = V_0 e^{\frac{-t}{RC}} = V_0 e^{\frac{-t}{\tau}}$$

Where $\tau = \text{RC}$ is the time constant of the ckt, which is the time required for the response to decay by a factor of 1/e or 36.8 percent of its initial value. This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called ***the natural response of the circuit***.

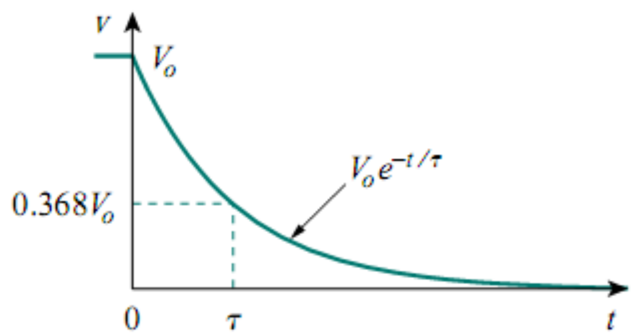


Fig. 3.1.2 The voltage response of the RC circuit.

THE SOURCE-FREE RL CIRCUIT

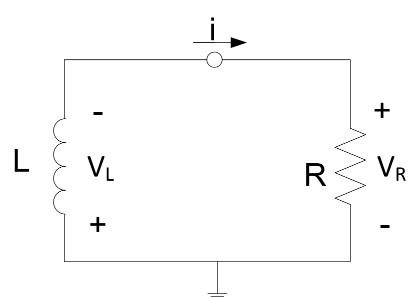


Fig. 3.1.3. A source free RL ckt

Our goal is to determine the circuit response, which we will assume to be the current i(t) through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At t = 0, we assume that the inductor has an initial current I_0, or

$$i(0) = I_0$$

Applying KVL around the loop in Fig.3.1.3,

$$v_L + v_R = 0$$

But v_L = Ldi/dt and v_R = iR. Thus,

$$L\frac{di}{dt} + Ri = 0$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt \quad \Longrightarrow \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$i(t) = I_0 e^{\frac{-Rt}{L}} = I_0 e^{\frac{-t}{\tau}}$$ Where, $\tau = \frac{L}{R}$ is the time constant of the ckt

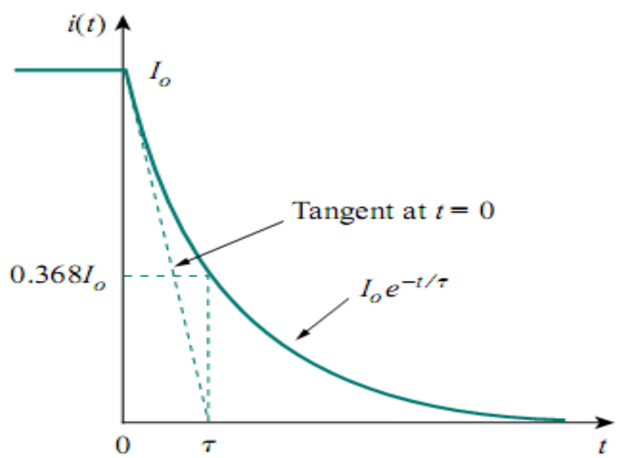


Fig. 3.1.4. The current response of the RL circuit

3.2 STEP RESPONSE OF AN RC & RL CIRCUITS AND SOLUTIONS

The step response is the response of the circuit due to a sudden application of a dc voltage or current source. The step response o fan RC or RL circuit has two components. Thus, we may write the response (x) as:

$$x = x_n + x_f$$

Where,

x_n _ is the natural response of the circuit, Since this part of the response will decay To almost zero after five time constants, it is also called the ***transient Response*** because it is a temporary response that will die out with time.

x_f _is known as the **forced response** because it is produced by the circuit when an External "force" is applied (a voltage source or current source). It represents What the circuit is forced to do by the input excitation. It is also known as the **Steady-state response**, because it remains a long time after the circuit is Excited.

(I) The differential equation approach

Solution of first order differential equation:

Consider the first order linear ordinary non-homogenous and homogenous differential equations:

$$\frac{dX(t)}{dt} + aX(t) = b \text{...} 3.2$$

$$\frac{dX(t)}{dt} + aX(t) = 0 \text{...} 3.3$$

The question here is to find $X(t)$ that satisfies equation 3.2.

Theorem: if $X(t) = X_p\,(t)$ is any solution of Eqn 3.2 (non homogenous eqn) and $X(t) = X_c\,(t)$ is any solution of the homogenous differential eqn 3.3 then

$$X(t) = X_p\,(t) + X_c\,(t) \text{...} 3.4$$

Where:

$X_p(t)$ is particular solution (forced solution)
$X_c(t)$ is complementary solution (natural solution)

Particular solution:

What is the function $X_p(t)$ such that if its differential is summed to a times $X_p(t)$ will give a constant b? If they are to sum up to a constant, both terms must be constant. A derivative of a given function is constant if the function is a linear function but the term $aX_p(t)$ would not be constant. This will force us to make $X_p(t)$itself constant as the ultimate solution.

$$X_p(t) = K1$$

Eqn 3.2 becomes

$$\frac{dK1}{dt} + aK1 = b$$

$$K1 = b/a$$

$$X_p(t) = K1 = b/a \text{...} 3.5$$

Complementary solution:

Just solve eqn 3.3

$$\left[\frac{dX_c(t)}{dt}\right]\left[\frac{1}{X_c(t)}\right] = -a$$

From your basic calculus we have $\frac{d[\ln(X_c(t))]}{dx} = \left[\frac{1}{X_c(t)}\right]\left[\frac{dX_c(t)}{dt}\right]$

So, we have $\left[\frac{dX_c(t)}{dt}\right]\left[\frac{1}{X_c(t)}\right] = \frac{d[\ln{(X_c(t))}]}{dx} = -a$

$$\ln{(X_c(t))} = \int -a\ dt$$

$$\ln{(X_c(t))} = -at + C$$

$$X_c(t) = e^{-at+C}$$

$$X_c(t) = e^{-at}e^C$$

Let us define another constant, $K_2 = e^C$

$$X_c(t) = K_2 e^{-at} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.6$$

The overall solution:

$$X(t) = X_p\,(t) + X_c\,(t)$$

$$X(t) = K1 + K_2 e^{-at}$$

This solution would bear a complete meaning by reminding an additional quantity.

Time constant: as we have seen earlier, is a parameter that determines the rate of decrease of $X(t)$ and given as follows.

$$\tau = 1/a$$

So finally we have,

$$X(t) = K1 + K_2 e^{-t/\tau} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.7$$

Example 3.2.1

For the network given in fig 3.2.1 below find $i_L(t)\ for\ t > 0$

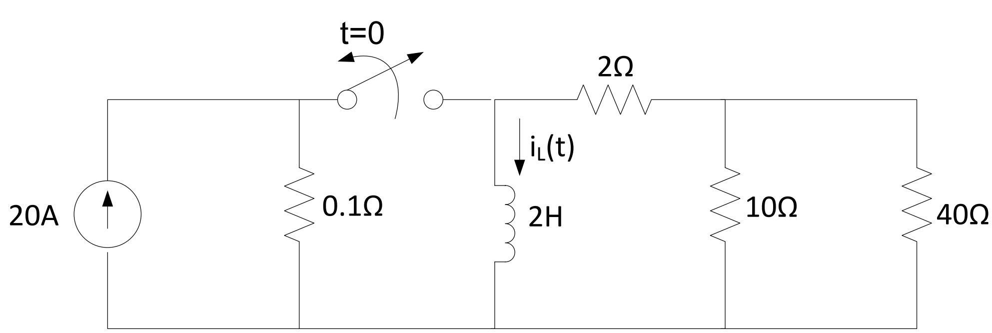


Fig 3.2.1

I. Find the initial condition. In this case we are concerned with the initial value of the current through the inductor. $i_L(0^-)$.

At $t = 0^-$ the inductor behaves as short circuit since the source is DC. The resulting equivalent circuit is shown here.

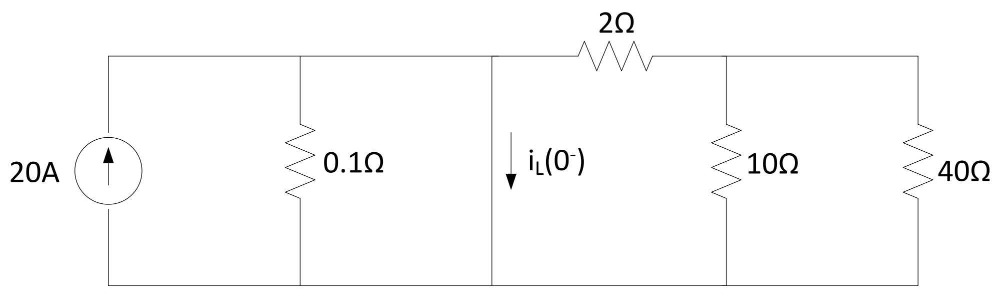


Fig 3.2.2 the inductor acts as short circuit

It can easily be seen from the figure that all of the current from the source will flow through the inductor. So, $i_L(0^-) = 20A$.

Remember the situation discussed in chapter one concerning the behavior of a current through an inductor. One of those lines reads 'the current through an inductor cannot change instantaneously. This means, $i_L(0^+) = 20A$.

II. At $t > 0$ the switch is open and our circuit will reduce to one shown in fig 3.2.3 (a). Clearly it is not in the standard format and we need to transform it. The circuit in fig (b) is obtained by combining the resistors in to one equivalent resistor.

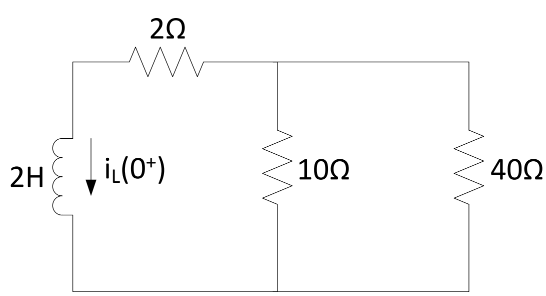


Fig3.2.3 (a) source is detached by the action The switch

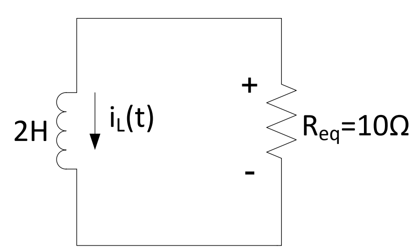


Fig3.2.3 (b) resistors replaced with of equivalent value

Note that,

$$R_{eq} = (10\Omega\,||40\Omega) + 2\Omega = 10\Omega$$

III. Taking KVL around the loop in fig 3.2.3(b) we have,

$$V_L(t) + R_{eq} i_L(t) = 0$$

$$L\frac{di_L(t)}{dt} + R_{eq} i_L(t) = 0$$

Replacing all values we have, $\frac{di_L(t)}{dt} + 5i_L(t) = 0$

IV. Finally,

$$i_L(t) = K_2 e^{-at}$$

From step three we can see that $a = 5\ hence\ ,\ \ i_L(t) = K_2 e^{-5t}$

K_2 could be obtained from the initial condition.

$$i_L(0) = 20A = K_2 e^{-5(0)}$$

$$K_2 = 20A$$

$$i_L(t) = 20e^{-at}$$

II. STEP BY STEP APPROACH

The step by step approach is somehow the faster way to reach on the solution and could be summarized in to the following steps.

I. Assume the solution is $X(t) = K_1 + K_2 e^{-t/\tau}$ .what else would it be?

II. Assume that the circuit is in steady state before the switch moves. This means we should replace capacitors by open circuit and inductors by short circuit. Then find $V_c(0^-)\ and\ i_L(0^-)$.

III. Now the switch is moved. Rearrange your circuit and do the following three.
- Replace a capacitor by voltage source = $V_c(0^-)$.
- Replace an inductor by a current source=$i_L(0^-)$.
- And take a moment to solve for $X(0)$.

IV. Assume$t = \infty\ and\ find\ X(t = \infty) = X(\infty)$ by replacing capacitor by open circuit and inductor by short circuit.

V. Find the time constant. The thumb rule for doing this is to find the Thevenin equivalent resistance w.r.t the terminals of the capacitor and the inductor. $\tau = R_{TH}C\ or\ \tau = L/R_{TH}$.

VI. Find the constants by using,

$$K_1 = X(\infty)$$

$$K_1 + K_2 = X(0)$$

$$K_2 = X(0) - X(\infty)$$

Finally, $$X(t) = X(\infty) + [X(0) - X(\infty)]e^{-t/\tau}$$

Example 3.2.2 find $V_0(t)$ as indicated in the fig below.

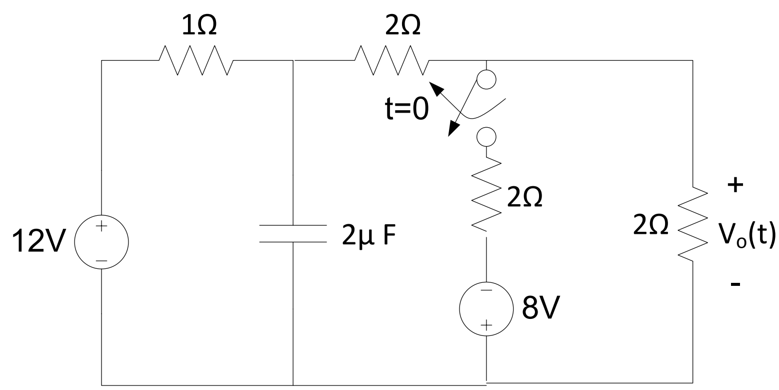


Fig 3.2.4

I. $V_0(t) = V_0(\infty) + [V_0(0) - V_0(\infty)]e^{-t/\tau}$

II. Assume steady state and replace capacitor by open circuit.

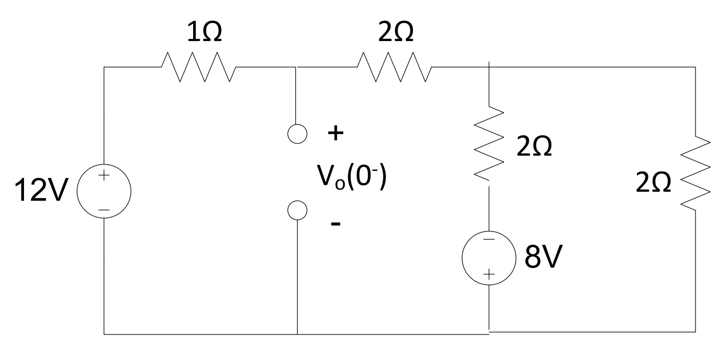


Fig 3.2.5

Take mesh around the two loops.
$-12 + 3i_1 + 2(i_1 - i_2) - 8 = 0$……………………..i
$2i_2 + 8 + 2(i_2 - i_1) = 0$……………………………ii
Obtaining, $V_c(0^-) = 12 - (1\Omega)i_1 = 12 - 1(4A) = 8V$

III. The switch is moved now $t = 0$. Replace the capacitor with voltage source of 8V. Note that the 8V source and the 2Ω are cut out of the network. Now, $V_0(0) = V_c(0^-)[\frac{2}{2+2}] = 4V$

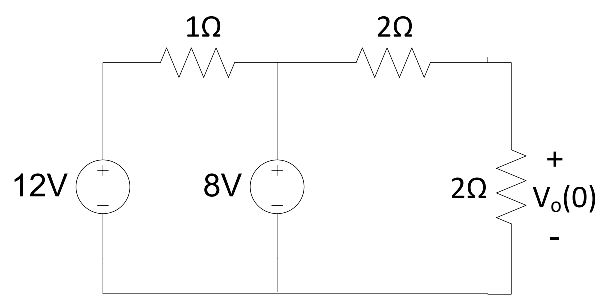


Fig 3.2.6

IV. At $t = \infty$ replace the capacitor by open circuit. So that,

$$V_0(\infty) = 12V[\frac{2}{2+2+1}] = \frac{24}{5}V$$

V. Find the time constant. Find the thevenin equivalent resistance w.r.t to x,y terminals as shown below.

$$R_{TH} = 1||(2+2) = \frac{4}{5}\Omega$$
$$\tau = R_{TH}C = \frac{4}{5}\Omega(2\text{F}) = \frac{8}{5}$$

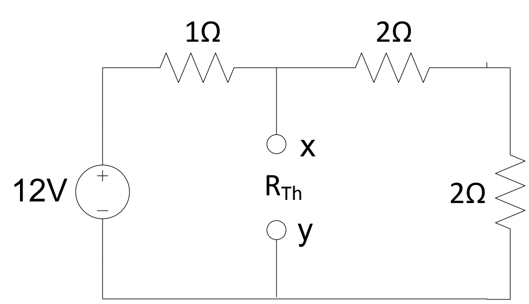


Fig 3.2.7 Thevenin equivalent across the terminals of the capacitor.

VI. $V_0(t) = V_0(\infty) + [V_0(0) - V_0(\infty)]e^{-t/\tau}$

$$V_0(t) = \frac{24}{5} + \left[4 - \frac{24}{5}\right]e^{-\frac{5}{8}t}$$

$$= \frac{24}{5} - \frac{4}{5}e^{-\frac{5}{8}t}$$

3.3. Second order transient circuits

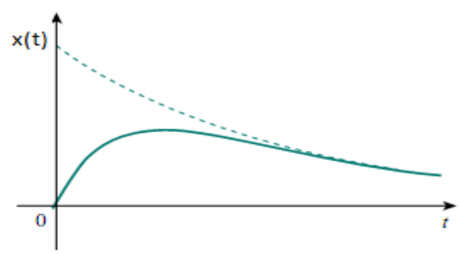
A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements. Given a second-order circuit, we can determine its step response x(t) (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value$x(\infty)$.

2. We find the natural response $x_n(t)$ by turning off independent Sources and applying KCL and KVL. Once a second-order differential equation is obtained in the form of $A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx=0$, by setting $s=\frac{dx}{dt}$ the characteristic equation would be $As^2+Bs+c=0$. Then determine its characteristic roots as $\left(s_{1,2}=-\alpha\pm\sqrt{\alpha^2-w_0}\right)$ Depending on whether the response is over damped, critically damped, or under damped, we obtain $x_n(t)$ with two unknown constants.

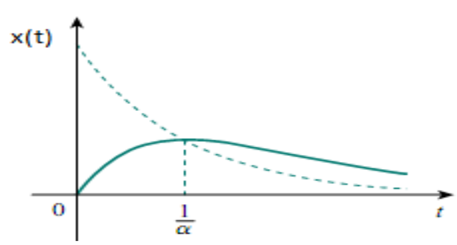
<u>Over damped Case (α>ω$_0$)</u>

$$x_n(t)=Ae^{s_1t}+Be^{s_2t}$$



<u>Critically Damped Case (α = ω$_0$)</u>

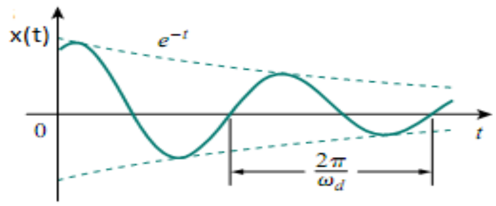
$$x_n(t)=(A+Bt)e^{-\alpha t}$$



<u>Under damped Case (α<ω$_0$)</u>

$$x_n(t)=e^{-\alpha t}(Acosw_dt+Bsinw_dt)$$

$$where\ \ w_d=\sqrt{w_0{}^2-\alpha^2}$$



3. We obtain the forced response as

$$x_f(t) = x(\infty)$$

Where $x(\infty)$ is the final value of x , obtained in step1.

4. The total response is now found as the sum of the natural response and forced response

$$x(t) = x_n(t) + x_f(t)$$

We finally determine the constants associated with the natural response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step1.
We can apply this general procedure to find the step response of any second-order circuit.

EXAMPLE 3.1

Find the complete response v and then i for t > 0 in the circuit of Fig.3.3.1.

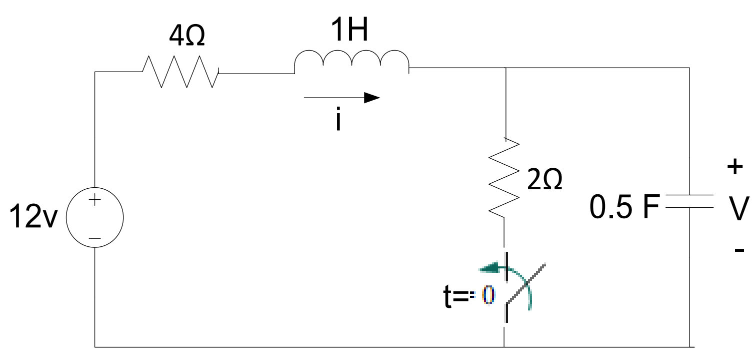


Fig. 3.3.1.

Solution:

Step 1

We first find the initial and final values. At t = 0−, the circuit is at stead state. The switch is open, the equivalent circuit is shown in Fig.3.3.2 (a).

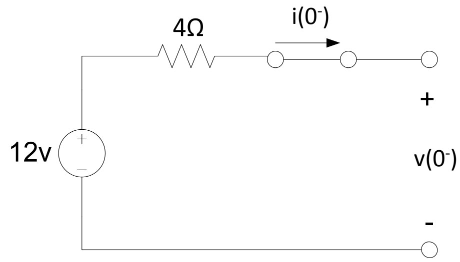


Fig.3.3.2 (a).

It is evident from the figure that, $v(0^-) = 12v, \quad i(0^-) = 0$
At t = 0^+,the switch is closed; the equivalent circuit is in Fig.3.3.2(b).

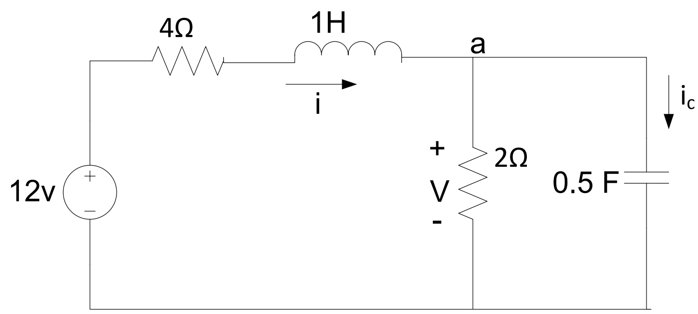


Fig.3.3.2(b).

By the continuity of capacitor voltage and inductor current, we know that

$$v(0^+) = v(0^-) = 12\text{V}, \quad i(0^+) = i(0^-) = 0 \quad \text{------------------------- (1)}$$

To get $dv(0^+)/dt$, we use $Cdv/dt = i_C$ or $dv/dt = i_C/C$. Applying KCL at node a in Fig.3.3.2(b),

$$i(0^+) = i_c(0^+) + \frac{v(0^+)}{2}$$

$$i_c(0^+) = -6A$$

Hence,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12\ v/s \text{ --- (2)}$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit inFig.3.3.2(b), giving

$$i(\infty) = \frac{12}{4+2} = 2A, \qquad v(\infty) = 2i(\infty) = 4v \text{ ------------------------------------- (3)}$$

Step 2

Next, we obtain the natural response for t > 0. By turning of f the 12-V voltage source, we have the circuit in Fig.3.3.3.

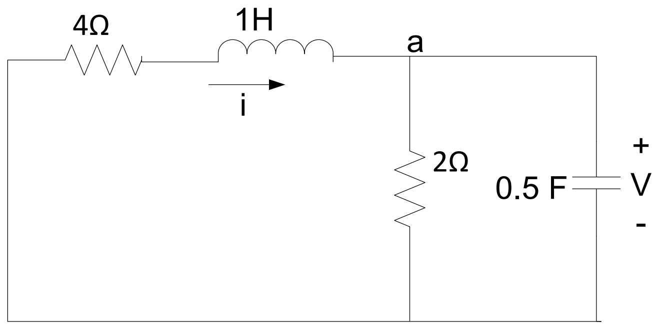


Fig.3.3.3

Applying KCL at node a in Fig.3.3.3 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} -(4)$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0 - (5)$$

Since we are interested in v for the moment, we substitute i from Eq.4 in to Eq.5.We obtain

$$\frac{d^2v}{dt^2} + \frac{5dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$$s^2 + 5s + 6 = 0$$

With roots s =−2 and s =−3.Thus,the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t} \text{ --- } (6)$$

where A and B are unknown constants to be determined later.

Step 3
The forced response is

$$v_f(t) = v(\infty) = 4 \text{ --- } (7)$$

Step 4
The complete response is

$$v(t) = v_n + v_f = 4 + Ae^{-2t} + Be^{-3t} \text{ -- } (8)$$

We now determine A and B using the initial values. From Eq.(1), v(0) = 12. Substituting this in to Eq.(8) at t = 0 gives

12 = 4 + A + B ⇒ A + B = 8 ---------------------------------- (9)

Taking the derivative of v in Eq.(8),

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t} \text{ --- (10)}$$

Substituting Eq.(2) into Eq.(10) at t = 0 gives

−12 =−2A − 3B ⇒ 2A + 3B = 12 ---------------------------- (11)

From Eqs.(9) and (11), we obtain

A = 12, B =−4

So that Eq.(8) becomes

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}, \quad t > 0 \text{ ------------------------ (12)}$$

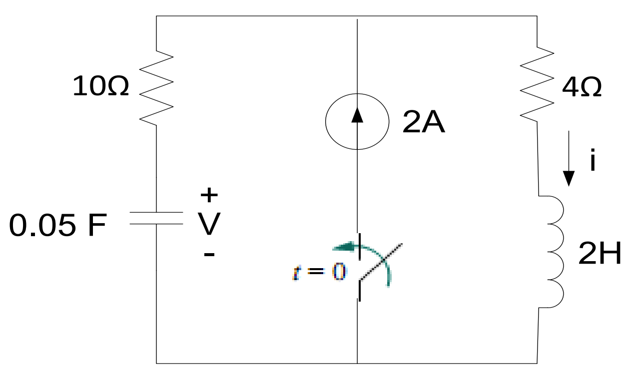
From v, we can obtain i,

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}$$

$$= 2 - 6e^{-2t} + 4e^{-3t}, \qquad t > 0$$

EXERCISE 1

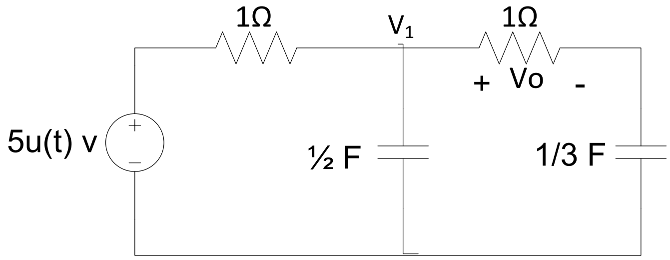
Determine v and i for t > 0 in the circuit shown below.



ANSWER: $8(1 - e^{-5t})V, \qquad 2(1 - e^{-5t})A$

EXERCISE 2

For t> 0, obtain $v_o(t)$ in the circuit of Fig. below. (Hint: First find v1 and v2.)



ANSWER: $2(e^{-t} - e^{-6t})\,V,\;\; t > 0$